UDC 539.3

INTERACTION OF A MOVING ELASTIC STAMP WITH AN ELASTIC HALF-PLANE THROUGH A STIFFENER OR A THIN IDEAL FLUID LAYER

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There are considered two contact problems for an elastic half-plane (plane strain case), 1) reinforced along its whole boundary by a thin elastic stiffener (stringer), and 2) coated by a thin ideal incompressible fluid layer. It is assumed that the stamp is impressed into the stiffener boundary or the fluid layer and moves at a constant velocity along this boundary. We neglect friction forces in the contact domain, and mass forces. By using a Fourier integral transformation, the problems are reduced to integral equations of the first kind with singular difference kernels. The structure of the solution of these equations is investigated. Asymptotic methods are used to construct the approximate solutions.

1. Problem of impressing a moving elastic stamp in an elastic half-plane reinforced along a stringer boundary. Let the elastic half-plane $(y \leqslant 0)$ with mechanical characteristics G_2, ν_2 and density ρ_2 be reinforced along its whole boundary $y = 0$ by a thin elastic stiffener, whose tensile deformation is described by the equation

$$ku'' = \tau_+ - \tau_- + \rho u^{\cdot\cdot}, \quad k = hE \tag{1.1}$$

Here h is the stiffener thickness, E is its Young's modulus, u is the mean displacement in the thickness along the x axis, τ_+, τ_- are tangent stresses acting along the upper and lower faces of the stiffener, respectively, and ρ is the density of the stiffener material. We assume that the stiffener is so thin that its resistance to bending strains can be neglected.

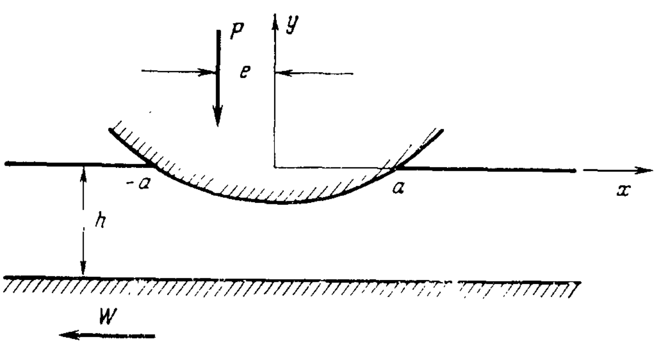

Fig. 1

An elastic stamp with the mechanical characteristics G_1, ν_1 moves at a constant velocity W along the upper stiffener boundary in the positive x direction. The stamp is pressed to the stiffener boundary by a force P, whereupon a contact domain of width $2a$ with the stiffener is formed. Outside the contact domain the stiffener upper boundary is stress-free.

Let us couple a moving coordinate system to the stamp Fig. 1.

$$x_1 = x - Wt, \quad y_1 = y \tag{1.2}$$

Let the stamp boundary be described by the equation $y = \psi(x)$ in this system, and let an unknown contact pressure $p(x)$ act on the stamp along the line of contact $-a \leqslant x \leqslant a$ (here and henceforth we omit the subscript one on x and y). Assuming the minimal radius of curvature of the stamp is $\rho_{\min} \gg a$ for $|x| < a$, we have the following expression (in the Hertz theory approximation)

$$v_1'(x, 0) = \frac{1}{\pi\theta_1} \int_{-a}^{a} \frac{p(\xi)}{\xi - x} d\xi, \quad \theta_1 = \frac{G_1}{1 - \nu_1} \tag{1.3}$$

for the derivative of the displacement v_1 of points of the stamp surface along the y axis.

The conditions for stiffener contact with the stamp and the elastic half-plane can be represented in the form

$$\tau_+ = 0, \quad \tau_- = \tau_{xy2}(x, 0), \quad u + u_2(x, 0) = 0 \tag{1.4}$$

Here $\tau_{xy2}(x, 0)$ is the tangential contact stress acting on the interface between the stiffener and the half-plane, and u_2 is the displacement of points of the half-plane along the x axis. The following relationship hence holds in the moving coordinate system because of (1.1).

$$c^2 u'' = -\tau_-(x), \quad c^2 u_2''(x, 0) = \tau_{xy2}(x, 0), \quad c^2 = k - \rho W^2 \tag{1.5}$$

Let us assume that $W < \sqrt{k\rho^{-1}}$.

Since the stiffener does not resist bending, we will still have on the half-plane boundary

$$\sigma_{y2}(x, 0) = p(x) \;\; (|x| \leqslant a), \quad \sigma_{y2}(x, 0) = 0 \;\; (|x| > a) \tag{1.6}$$

Let us admit that the velocity of stamp motion along the stiffener boundary is less than the compression and shear strain wave propagation velocities in the half-plane, hence

$$\beta^2 = 1 - \frac{1 - 2\nu_2}{2(1 - \nu_2)} V^2 > 0, \quad \gamma^2 = 1 - V^2 > 0, \quad V = W\sqrt{\frac{\rho_2}{G_2}} \tag{1.7}$$

By using the Fourier integral transform in the variable x we now find the solution of the Lamé equations with inertial terms with the boundary conditions (1.5) and (1.6) in the half-plane and the condition of decreasing stress at infinity. We hence obtain

$$v_2'(x, 0) = -\frac{1}{\pi\theta_2\mu} \int_{-a}^{a} p(\xi)\, d\xi \int_0^\infty \frac{u + D}{u + 1} \sin\frac{u(\xi - x)}{\mu}\, du \tag{1.8}$$

$$\theta_2 = \frac{G_2\gamma(1-\gamma^2)}{1-\beta\gamma}, \quad \mu = \frac{c^2\gamma(1-\gamma^2)}{G_2[4\beta\gamma-(1+\gamma^2)^2]}$$

$$D = \frac{\beta\gamma(1-\gamma^2)^2}{(1-\beta\gamma)[4\beta\gamma-(1+\gamma^2)^2]}$$

Again taking into account that the stiffener does not resist bending, we write the condition for the relation between $v_1'(x, 0)$ and $v_2'(x, 0)$ which holds on the contact line

$$v_1'(x, 0) - v_2'(x, 0) = \varepsilon - \psi'(x) \qquad (|x| \leqslant a) \tag{1.9}$$

Here ε is the angle of stamp rotation upon insertion into the stiffener.

Substituting (1.3) and (1.8) into (1.9), we obtain an integral equation to determine the contact pressure $p(x)$

$$\int_{-a}^{a} \frac{p(\xi)}{\xi - x} d\xi + \frac{\sigma}{\mu} \int_{-a}^{a} p(\xi) H\left(\frac{\xi - x}{\mu}\right) d\xi = \pi\theta_{12}[\varepsilon - \psi'(x)] \tag{1.10}$$

$$\sigma = \frac{(D-1)\theta_1}{\theta_1 + \theta_2}, \quad \theta_{12} = \frac{\theta_1\theta_2}{\theta_1 + \theta_2}, \quad H(t) = \int_0^\infty \frac{\sin ut}{u+1} du$$

which can also be represented in the form

$$\int_{-a}^{a} p(\xi) K\left(\frac{\xi - x}{\mu}\right) d\xi = \pi\mu\theta_{12}[\varepsilon - \psi'(x)], \tag{1.11}$$

$$K(t) = \int_0^\infty \frac{u + \sigma + 1}{u + 1} \sin ut \, du$$

The statics conditions

$$P = \int_{-a}^{a} p(\xi) d\xi, \quad Pe = \int_{-a}^{a} \xi p(\xi) d\xi \tag{1.12}$$

must still be appended to (1.10) or (1.11). Here e is the eccentricity of the application of the impressing force P. Let us note that the force P should be so large that the condition $p(x) > 0$ would be satisfied for all $|x| < a$. The second condition in (1.12) sets up the interrelation between ε and e.

Three cases can be represented in solving (1.10) or (1.11) together with the first condition in (1.12):

a) If $\psi'(\pm a \mp 0) - \psi'(\pm a, \pm 0) = \text{const} \neq 0$, then the quantity a should be considered given, and the function $p(x)$ can have an integrable singularity at the points $x = \pm a$;

b) If the derivative of the function $\psi(x)$ is continuous in the neighborhood of the point $x = a$ (or $x = -a$), then the quantity a is to be determined, and the function $p(x)$ should be bounded at the point $x = a$;

c) If the derivative of the function $\psi(x)$ is continous in the neighborhood of the points $x = \pm a$, then the quantity a is also to be determined, but the function $p(x)$

should be bounded at the points $x = \pm a$.

Let us note that the problem posed has been examined earlier in a particular case in [1].

2. Problem on the gliding of an elastic body over the wet boundary of an elastic half-plane. Let an elastic half-plane G_2, ν_2, ρ_2, $y \leqslant 0$ be covered by an ideal incompressible fluid of thickness h, and let the fluid density be ρ. An elastic body (G_1, ν_1) pressed to the layer boundary by a force P (Fig. 1) glides at a constant velocity W over the upper boundary of the layer. Let us connect a moving coordinate system (1.2) to the body. In this system e is the action arm of the force, and the length of a section of the body boundary which is in contact with the fluid is determined by the inequalities $-a \leqslant x \leqslant a$, $2a \gg h$. The pressure p on the fluid surface equals zero outside the contact section. Let us assume that the flow is potential and stationary. The boundary of the gliding body is described by the equation $y = -\delta - \varepsilon x + \psi(x)$, where $p > 0$ and $y \gg -h$ for all $|x| \leqslant a$ and the minimal radius of body curvature is $\rho_{\min} \gg a$ for all $|x| \leqslant a - \varepsilon, \varepsilon > 0$.

Taking account of the assumptions made, we linearize the Bernoulli integral in the neighborhood of the main flow, for which $v_{x0} = W$, $v_{y0} = 0$, $p_0 = 0$. The contact pressure $p(x)$ acting between the gliding body and the fluid for $|x| \leqslant a$ and also a, ε and e are the quantities desired in the solution of the problem.

We have the equation

$$\Delta\varphi = 0, \quad v_x = W + \frac{\partial\varphi}{\partial x}, \quad v_y = \frac{\partial\varphi}{\partial y}, \quad p = -\rho W\frac{\partial\varphi}{\partial x} \tag{2.1}$$

Here and above v_x, v_y are the velocity projections in the x and y axis directions, and $\varphi(x, y)$ is the velocity potential.

Let us introduce the dimensionless variables

$$y' = y/h, \quad x' = x/a, \quad \sigma = h/a \ll 1 \tag{2.2}$$

and we easily see that the degenerate flow, for very small σ, in the fluid layer is described by the equation $\varphi_{y'}'' = 0$. Solving this equation and returning to the old variables, we find

$$\begin{gathered}\varphi(x, y) = F_1(x)\, y + F_2(x), \quad v_y = F_1(x), \\ p = -\rho W\,[F_1'(x) y + F_2'(x)]\end{gathered} \tag{2.3}$$

Now, we note that a pressure $p(x)$ acts at $y = 0$ and $|x| \leqslant a$ on the fluid from the elastic body, and the pressure $q(x)$ at $y = -h$ and $|x| < \infty$ from the elastic half-plane. For $y = 0$ and $|x| > a$ we have $p(x) = 0$ by definition. Satisfying these conditions and taking still into account that $q = v_y = 0$ for $x = -\infty$, we have

$$v_y = F_1(x) = \frac{1}{\rho W h}\Big[\int_{-\infty}^{x} p^*(\xi)\,d\xi - \int_{-\infty}^{x} q(\xi)\,d\xi\Big] \tag{2.4}$$

Here $p^*(x) = p(x)$ for $|x| \leqslant a$, $p^*(x) = 0$ for $|x| > a$.

After the customary linearization used in the theory of a thin slightly cambered wing, we obtain the following condition on the contact boundaries of the fluid and the elastic body and the elastic half-plane:

$$v_y\big|_\Gamma = Wf'(x) \tag{2.5}$$

where $y = f(x)$ is the equation of the boundary. Therefore, we have the following contact condition with the elastic body by virtue of (2.5)

$$F_1(x) = [v_1'(x, 0) + \psi'(x) - \varepsilon]W \quad (|x| \leqslant a) \tag{2.6}$$

and with the elastic half-plane

$$F_1(x) = v_2'(x, -h)\, W \quad (|x| < \infty) \tag{2.7}$$

Here v_1 and v_2 are elastic displacements of points of the body and the half-plane along the y axis. The quantity $v_1'(x, 0)$ is given by (1.3).

To determine the quantity $v_2'(x, -h)$, we subject the Lamé equations with the inertia terms to a Fourier integral transformation in the variable x and we construct their solution for a half-plane under the boundary conditions

$$\sigma_y(x, -h) = -q(x), \quad \tau_{xy}(x, -h) = 0 \quad (|x| < \infty) \tag{2.8}$$

and the condition of no stresses at infinity. We also take into account here that $q(x) \to 0$ as $|x| \to \infty$ and that the inequalities (1.7) hold. We consequently obtain

$$v_2'(x, -h) = \frac{i}{2\pi\theta_2} \int_{-\infty}^{\infty} \operatorname{sgn} \alpha Q(\alpha) e^{-i\alpha x} d\alpha \tag{2.9}$$

$$\theta_2 = G_2 [4\gamma\beta - (1 + \gamma^2)^2]\, \beta^{-1} (1 - \gamma^2)^{-1} \tag{2.10}$$

Here $Q(\alpha)$ is the Fourier transform of the function $q(x)$.

Substituting (2.4) and (2.9) into (2.7), and then subjecting this relationship to Fourier transformation, we set up a connection between $Q(\alpha)$ and the Fourier transform $P(\alpha)$ of the function $p^*(x)$

$$Q(\alpha) = \frac{P(\alpha)}{\mu|\alpha| + 1}, \quad \mu = \frac{\rho h W^2}{\theta_2} \tag{2.11}$$

The inverse transformation yields

$$q(x) = \frac{1}{\pi\mu} \int_{-a}^{a} p(\xi)\, d\xi \int_0^\infty \frac{\cos ut}{u + 1} du \left(t = \frac{\xi - x}{\mu}\right) \tag{2.12}$$

An integral equation in $p(x)$ can now be obtained if the contact condition (2.6) is transformed with (2.4), (1.3) and (2.12) taken into account. We will have

$$\int_{-a}^{a} \frac{p(\xi)}{\xi - x} d\xi + \frac{\sigma}{\mu} \int_{-a}^{a} p(\xi) H\left(\frac{\xi - x}{\mu}\right) d\xi = \pi\theta_1 [\varepsilon - \psi(x)] \tag{2.13}$$

$$(|x| \leqslant a, \ \sigma = \theta_1\theta_2^{-1})$$

where $H(t)$ is given by (1.10). Equation (2.13) can also be represented in the form of (1.11), where θ_{12} should be replaced by θ_1. Condition (1.12) must also be appended to (2.13).

Furthermore, let us examine two possible gliding modes:

a) If $\psi'(a-0) - \psi'(a+0) = \text{const} \neq 0$, then the quantity a should be considered known and a solution of (2.13) should exist such that the quantity $p(a)$ is bounded (Joukowski's condition).

b) If the derivatives $\psi'(x)$ and $\psi''(x)$ are continuous in the neighborhood of the point $x = a$, then the quantity a is to be determined and $p(a)$ and $p'(a)$ should be bounded [2].

Therefore, both problems under consideration are reduced to the same integral equation of the form (1.10) or (1.11) ($\theta_{12} = \theta_1$ for the second problem). The constants σ, μ and θ_2 in the integral equation are given by (1.8) and (1.10) for the first problem and by (2.10), (2.11) and (2.13) for the second.

3. Structure of the solutions of the integral equation (1.10), asymptotic solutions. Evaluating the integral [3], we represent the kernel $H(t)$ of the integral equation (1.10) in the form

$$H(t) = (\operatorname{ci} t \sin|t| - \cos t \operatorname{si}|t|) \operatorname{sgn} t \tag{3.1}$$

Here $\operatorname{ci} t$ and $\operatorname{si} t$ are the integral cosine and sine. Using the known expansions of these functions in series for small t [3], we see that for $H(t)$ the following representation is valid for all $0 \leqslant |t| < \infty$:

$$H(t) = \ln|t|\, H_1(t) + |t|\, H_2(t) + H_3(t) + {}^1/_2\pi \operatorname{sgn} t \tag{3.2}$$

$$H_i(t) = \sum_{k=0}^{\infty} a_{ik} t^{2k+1} \quad (i = 1, 2, 3)$$

$$a_{1k} = \frac{(-1)^k}{(2k+1)!}, \quad a_{2k} = \frac{\pi(-1)^{k+1}}{2(2k+2)!}, \quad a_{30} = C - 1 \tag{3.3}$$

$$a_{3k} = (-1)^k \left\{\frac{1}{(2k+1)!}\left(C - \frac{1}{2k+1}\right) + \sum_{s=1}^{k} \frac{2k+1-4s}{2s(2s)!\,(2k-2s+1)(2k-2s+1)!}\right\}$$

Here C is the Euler constant, and $k \geqslant 1$ in the expression for a_{3k}.

Analyzing the representation (3.2), we conclude that for all $|t| \leqslant A$, where A is an arbitrary number as large as desired, the following relationship is valid:

$$H(t) = \frac{\pi}{2} \operatorname{sgn} t + t \ln|t| + F(t), \quad F(t) \in B_1^{\,1}(-A, A) \tag{3.4}$$

Here $B_k^{\,\alpha}(-A, A)$ is the space of functions whose k-th derivatives satisfy the Hölder condition with $0 < \alpha \leqslant 1$ for $t \in [-A, A]$.

The results of [4] can be used to prove the following theorems.

Theorem 1. Let $\psi(x) \in B_1^{\,\alpha}(-a, a)$, $0 < \alpha \leqslant 1$. Then if a solution of (1.10) exists for $\mu \in (0, \infty)$ such that $p(x) \in L_p(-a, a)$, $1 < p < 2$, then $p(x)$ has the form

$$p(x) = \omega(x)(a^2 - x^2)^{-1/2}, \quad \omega(x) \in B^{\gamma}(-a, a), \quad \gamma = \inf\left(\alpha, \frac{p-1}{p}\right)$$

Theorem 2. Let 1) $\psi(x) \in B_1^{\alpha}(-a, a)$, $0 < \alpha \leqslant 1$, 2) $\psi(x) \in B_1^{\beta}(a-\varepsilon, a)$, $\varepsilon > 0$, $1/2 < \beta \leqslant 1$. Then if a solution of (1.10) exists for a given $\mu \in (0, \infty)$ such that 1) $p(x) \in L_p(-a, a)$, $1 < p < 2$, 2) $|p(x)| \leqslant m, m > 0$ for $a - \varepsilon \leqslant x \leqslant a$, then the relationship

$$P = -\pi\theta_{12}\varepsilon a + \int_{-a}^{a} \sqrt{\frac{a+\xi}{a-\xi}} [\theta_{12}\psi'(\xi) + \sigma g(\xi)]\, d\xi \tag{3.5}$$

$$g(x) = \frac{1}{\pi\mu}\int_{-a}^{a} p(\xi) H\left(\frac{\xi - x}{\mu}\right) d\xi$$

is satisfied, and $p(x)$ has the form

$$p(x) = \sqrt{\frac{a-x}{a+x}}\, \omega(x), \quad \omega(x) \in B^{\gamma}(-a, a), \tag{3.6}$$

$$\gamma = \inf\left(\alpha, \frac{p-1}{p}, \beta - \frac{1}{2}\right)$$

Theorem 3. Let 1) $\psi(x) \in B_1^{\alpha}(-a, a)$, $0 < \alpha \leqslant 1$; 2) $\psi(x) \in B_1^{\beta}(a-\varepsilon, a)$, $\varepsilon > 0$, $1/2 < \beta \leqslant 1$, 3) $\psi(x) \in B_1^{\beta}(-a, -a+\varepsilon)$. Then, if a solution of (1.10) exists for $\mu \in (0, \infty)$ such that 1) $p(x) \in L_p(-a, a)$, $1 < p < 2$, 2) $|p(x)| \leqslant m, m > 0$ for $x \in [a-\varepsilon, a]$ and $x \in [-a, -a + \varepsilon]$, then the relationships

$$P = \int_{-a}^{a} [\theta_{12}\psi'(\xi) + \sigma g(\xi)] \frac{\xi\, d\xi}{\sqrt{a^2 - \xi^2}} \tag{3.7}$$

$$\pi\varepsilon\theta_{12} = \int_{-a}^{a} [\theta_{12}\psi'(\xi) + \sigma g(\xi)] \frac{d\xi}{\sqrt{a^2-\xi^2}}$$

are satisfied and $p(x)$ has the form

$$p(x) = \omega(x)\sqrt{a^2 - x^2}, \quad \omega(x) \in B^{\gamma}(-a, a) \tag{3.8}$$

$$\gamma = \inf\left(\alpha, \frac{p-1}{p}, \beta - \frac{1}{2}\right)$$

Theorem 4. Let 1) $\psi(x) \in B_1^{\alpha}(-a, a)$, $0 < \alpha \leqslant 1$, 2) $\psi(x) \in B_2^{\beta}(a-\varepsilon, a)$, $\varepsilon > 0$, $1/2 < \beta \leqslant 1$. Then if a solution of (1.10) exists for $\mu \in (0, \infty)$ such that 1) $p(x) \in L_p(-a, a)$, $1 < p < 2$, 2) $|p'(x)| \leqslant m$, $m > 0$ for $a - \varepsilon \leqslant x \leqslant a$, then the relationships (3.5) and

$$\pi\theta_{12}[\varepsilon - \psi'(a)] - \pi\sigma g(a) = \int_{-a}^{a} \frac{\sqrt{a+\xi}}{(a-\xi)^{3/2}} \{\theta_{12}[\psi'(a) - \psi'(\xi)] + \sigma[g(a) - g(\xi)]\}\, d\xi \tag{3.9}$$

are satisfied, and $p(x)$ has the form

$$p(x) = \frac{(a-x)^{3/2}}{\sqrt{a+x}}\, \omega(x), \quad \omega(x) \in B^{\gamma}(-a, a) \tag{3.10}$$

$$\gamma = \inf\left(\alpha, \frac{p-1}{p}, \beta - \frac{1}{2}\right)$$

As an illustration, let us present the proof of Theorem 4, taking into account that Theorems 1 — 3 are proved more simply and by an analogous scheme. We show that a function $g(x)$ of the form (3.5) possesses the following properties:

$$g(x) \in B^{\delta}(-a, a), \qquad \delta = \frac{p-1}{p}, \qquad g(x) \in B_1^{1-0}(a-\varepsilon, a) \tag{3.11}$$

To do this, we represent $g(x)$ by using (3.4) in the form

$$g(x) = g_1(x) + g_2(x) + g_3(x), \quad g_1(x) = \frac{1}{2\mu}\left(P - 2\int_a^x p(\xi)\,d\xi\right) \tag{3.12}$$

$$g_2(x) = \frac{1}{\pi\mu^2}\left[P(x-e)\ln\mu + \int_{-a}^{a} p(\xi)(\xi - x)\ln|\xi - x|\,d\xi\right]$$

$$g_3(x) = \frac{1}{\pi\mu}\int_{-a}^{a} p(\xi)F\left(\frac{\xi - x}{\mu}\right)d\xi$$

Because of the properties of the function $p(\xi)$ mentioned in the theorem, and the properties of $F(t)$ noted in (3.4), it can be shown that $g_3(x) \in B_1^1(-a, a)$. Let us differentiate $g_2(x)$ with respect to x and let us estimate

$$|g_2'(x_1) - g_2'(x_2)| \leqslant \frac{1}{\pi\mu^2}\int_{-a}^{a}|p(\xi)|\left|\ln\left|\frac{\xi - x_1}{\xi - x_2}\right|\right|d\xi \leqslant \tag{3.13}$$

$$\frac{1}{\pi\mu^2}\|p\|_{L_p}\left(\int_{-a}^{a}\left|\ln\left|\frac{\xi - x_1}{\xi - x_2}\right|\right|^q d\xi\right)^{1/q} \leqslant$$

$$\mu^{-2}M\|p\|_{L_p}|x_1 - x_2|^{1/q-0}, \qquad \frac{1}{q} = \frac{p-1}{p} = \delta$$

$$|g_2'(a) - g_2'(x)| \leqslant \frac{1}{\pi\mu^2}\int_{-a}^{a}|p(\xi)|\left|\ln\left|\frac{\xi - a}{\xi - x}\right|\right|d\xi \leqslant \frac{1}{\mu^2}m^*(a-x)^{1-0},$$

$$x > -a$$

It follows from (3.13) that $g_2(x) \in B_1^{\delta-0}(-a, a)$ and $g_2(x) \in B_1^{1-0}(a-\varepsilon, a)$. The following estimates hold for $g_1(x)$

$$|g_1(x_1) - g_1(x_2)| = \frac{1}{\mu}\left|\int_{x_1}^{x_2} p(\xi)\,d\xi\right| \leqslant \frac{1}{\mu}\|p\|_{L_p}|x_1 - x_2|^{1/q} \tag{3.14}$$

$$|g_1'(a) - g_1'(x)| = \frac{1}{\mu}|p(a) - p(x)| = \frac{1}{\mu}m(a-x), \qquad x > -a$$

It follows from (3.14) that $g_1(x) \in B^{\delta}(-a, a)$ and $g_1(x) \in B_1{}^1(a-\varepsilon, a)$. Therefore, the relationships (3.11) are proved.

We note that the integral equation (1.10) can be written in the form

$$\int_{-a}^{a}\frac{p(\xi)}{\xi - x}\,d\xi = \pi\{\theta_{12}[\varepsilon - \psi'(x)] - \sigma g(x)\}, \qquad (|x| \leqslant a) \tag{3.15}$$

We temporarily assume that the function $g(x)$ is known, then the solution of the singular integral equation of the first kind with Cauchy kernel (3.15), which possesses the properties mentioned in the conditions of Theorem 4, will have the form [5]

$$p(x)=\frac{1}{\pi}\frac{(a-x)^{3/2}}{\sqrt{a+x}}\int_{-a}^{a}\frac{\sqrt{a+t}}{(a-t)^{3/2}}\{\theta_{12}[\psi'(t)-\psi'(a)]+ \tag{3.16}$$

$$\sigma[g(t)-g(a)]\frac{dt}{t-x}$$

under the additional conditions (3.5) and (3.9). Here, the properties of the functions $\psi(x)$ mentioned in the conditions of Theorem 4, as well as the properties (3.11) of the function $g(x)$ are used essentially. Equation (3.16) can be rewritten as follows:

$$p(x)=\frac{(a-x)^{3/2}}{\sqrt{a+x}}\omega(x) \tag{3.17}$$

$$\omega(x)=\frac{1}{\pi}\int_{-a}^{a}\frac{h(t)}{t-x}dt-[\theta_{12}\psi''(a)+\sigma g'(a)]$$

$$h(t)=\frac{\sqrt{a+t}}{(a-t)^{3/2}}\{\theta_{12}[\psi'(t)-\psi'(a)+\psi''(a)(a-t)]+$$

$$\sigma[g(t)-g(a)+g'(a)(a-t)]\}$$

We note that $h(\pm a)=0$, and by virtue of (1.30) [4], the inequalities

$$\begin{aligned}&|\psi'(t)-\psi'(a)+\psi''(a)(a-t)|\leqslant A_1|t-a|^{\beta+1}\\&|g(t)-g(a)+g'(a)(a-t)|\leqslant A_2|t-a|^{2-0}\end{aligned} \tag{3.18}$$

hold in the neighborhood of the point $t=a$.

Now, taking account of (3.18) as well as the other properties of the functions $\psi(x)$ and $g(x)$ we arrive at the conclusion that $h(t)\in B^{\gamma}(-a,a)$, $\gamma=\inf(\alpha,\beta-{}^1/_2,\delta)$. Furthermore, reasoning for the integral (3.17) just as for the integral (2.3) [4], we see that the function $\omega(x)$ of the form (3.17) belongs to the class $B^{\gamma}(-a,a)$.

Let us turn to the problem under consideration. The condition (3.5) is used to determine the quantity a for variant b) of the first problem, the quantity a will be determined from the first condition of (3.7) for the variant c), and the second condition in (3.7) permits the determination of the quantities ε and e together with the second condition of (1.12). Together with the second condtion in (1.12), condition (3.5) permits the determination of the quantities ε and e for variant a) of the second problem, while the quantities a, ε and e for variant b) can be found from conditions (3.5), (3.9) and the second condition in (1.12).

As the expansion (3.2) shows, for large values of the parameter λ the solution of the integral equation (1.10) must be sought in the form [6]

$$p(x)=\sum_{i=0}^{\infty}\sum_{j=0}^{i}p_{ij}(x)\lambda^{-i}(\ln\lambda)^{j} \tag{3.19}$$

Substituting (3.2) and (3.19) into (1.10), inverting its singular part exactly, and then equating coefficients of identical powers of λ^{-1} and $\ln\lambda$ in the left and right sides of the relationship obtained, we obtain an infinite system of relations for the sequential determination of the functions $p_{ij}(x)$. Limiting ourselves to a finite number of terms in (3.19), we can obtain an asymptotic representation for $p(x)$ for large λ by the method mentioned. Using the Banach fixed-point principle a λ_0, $0 < \lambda_0 < \infty$ can be found such that for $\lambda > \lambda_0$ the solution of (1.10) in the class $L_p(-a, a)$, $1 < p < 2$ exists and is unique, and converges uniformly in λ in the norm $L_p(-a, a)$ in the double series (3.19).

We note that $0 \leqslant |t| \leqslant 2/\lambda$ in (3.2), and hence, for sufficiently large λ it is possible to take $H(t) = {}^1/_2\pi \operatorname{sgn} t$ approximately. In this case (1.10) can be given the form of the Prandtl integral equation for a finite-span wing

$$\int_{-a}^{a} \frac{\varphi'(\xi)}{\xi - x} d\xi = \frac{\pi\sigma}{\mu}\left[\varphi(x) - \frac{1}{2}P\right] + \pi\theta_{12}[\varepsilon - \psi'(x)] \qquad (|x| \leqslant a) \qquad (3.20)$$

$$\varphi(x) = \int_{-a}^{x} p(\xi)\,d\xi, \quad \psi(-a) = 0, \quad \varphi(a) = P$$

A whole arsenal of methods of finite-span wing theory, as well as the theory of elastic stiffeners (stringers), particularly the results in [7, 9], can be used to find approximate solutions of (3.20).

For very large λ when the expression in the first square brackets can be neglected in the right side of (3.20), the solution for variants a) – c) of the first problem has the form

Variant a)

$$p(x) = \frac{1}{\pi\sqrt{a^2 - x^2}}\left[P + \theta_{12}\left(\pi\varepsilon x + \int_{-a}^{a} \frac{\psi'(\xi)\sqrt{a^2 - \xi^2}}{\xi - x} d\xi\right)\right] \qquad (3.21)$$

$$Pe = \theta_{12}\left[\frac{\pi\varepsilon a^2}{2} - \int_{-a}^{a} \psi'(\xi)\sqrt{a^2 - \xi^2}\,d\xi\right] \qquad (3.22)$$

Variant b)

$$p(x) = \theta_{12}\sqrt{\frac{a - x}{a + x}}\left[-\varepsilon + \frac{1}{\pi}\int_{-a}^{a}\sqrt{\frac{a + \xi}{a - \xi}}\frac{\psi'(\xi)}{\xi - x} d\xi\right] \qquad (3.23)$$

(3.22) and (3.5) must be appended here for $g(x) \equiv 0$;

Variant c)

$$p(x) = \frac{\theta_{12}}{\pi}\sqrt{a^2 - x^2}\int_{-a}^{a} \frac{\psi'(\xi)\,d\xi}{\sqrt{a^2 - \xi^2}\,(\xi - x)} \qquad (3.24)$$

and (3.22) and (3.5) must also be appended here for $g(x) \equiv 0$.

The solution for variant a) of the second problem is given by (3.23), (3.22) and (3.5) for $g(x) \equiv 0$ with θ_{12} replaced by θ_1, while we have for variant b)

$$p(x) = \frac{\theta_1}{\pi}\frac{(a - x)^{3/2}}{\sqrt{a + x}}\int_{-a}^{a}\frac{\sqrt{a + \xi}}{(a - \xi)^{3/2}}\frac{[\psi'(a) - \psi'(\xi)]}{\xi - x} d\xi \qquad (3.25)$$

where (3.22), (3.5) and (3.9) must also be appended here for $g(x) \equiv 0$ with θ_{12} replaced by θ_1.

A certain modification of the method of "small λ " [6] permits the construction of the principal term of the asymptotic solution of (1.11) for the problems under consideration with small values of λ. The main difficulty lies in factorizing the function

$$L_\varepsilon(u) = \frac{u(\sqrt{u^2+\varepsilon^2}+\sigma+1)}{\sqrt{u^2+\varepsilon^2}(\sqrt{u^2+\varepsilon^2}+1)} = L_+(u)L_-(u) \quad (\varepsilon \to 0)$$

Let us henceforth limit ourselves to the construction of just the "degenerate" solution which is suitable for the case of very small λ.

We note that for large t the asymptotic representation

$$K(t) \sim (1+\sigma)t^{-1}$$

holds for a kernel $K(t)$ of the form (1.11), and the degenerate solution of the problems for very small λ will be determined by the equation

$$\int_{-a}^{a} \frac{p(\xi)}{\xi - x} d\xi = \pi\theta_{12}^*[\varepsilon - \psi'(x)] \quad (|x| \leqslant a)$$

where for the first and second problems, respectively

$$\theta_{12} = \frac{\theta_1\theta_2}{\theta_2 + D\theta_1}, \quad \theta_{12}^* = \frac{\theta_1\theta_2}{\theta_1+\theta_2}$$

Then the solutions of the problems for very small λ will be determined as before, by (3.21)−(3.25) and (3.5),(3.7),(3.9), where $g(x) \equiv 0$ and the quantity θ_{12} or θ_1 is replaced by the appropriate expression θ_{12}^*.

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